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A ring-source/integral-equation method for the calculation of hydrodynamic forces exerted on floating bodies of revolution

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The wave forces exerted on a floating 3-dimensional body can be found by expressing the velocity potential of the surrounding fluid as the field of a distribution of point wave sources over the wetted part of the body surface. The problem then reduces to one of finding the solution to a 2-dimensional Fredholm integral equation of the second kind, to give the (unknown) surface source density. A simplification is possible for bodies that have a vertical axis of symmetry: for this type of body we can distribute 'rings of sources' over the body surface, and the problem then reduces to the solution of 1-dimensional Fredholm equations of the second kind. This approach has been adopted before, but earlier work has made use of expressions for the fundamental ring-source potentials which are not always suitable for numerical computation. It is possible to derive many alternative expressions for the ring-source potentials, but it appears that no single expression is computationally convenient in every situation; the present paper discusses the computational merits of three different types of expression, the aim being to provide a comprehensive scheme for the evaluation of the ring-source potentials. The ring-source/integral-equation method will be used to calculate the wave forces exerted on certain specific bodies of revolution and results are presented here. A brief discussion of the problem of 'irregular values' is also given: these only occur when the body intersects the free surface.

1. Introduction

The subject of the radiation or diffraction of water waves by natural boundaries or man-made structures is of considerable importance in ocean engineering. This has been especially true of the last 20 years where the discovery of sizeable oil deposits under the North Sea has prompted detailed research into the safe and economic design of offshore terminals and drilling platforms. Even more topical than this is the current debate over the practicability of placing wave-energy extraction devices off the shores of the United Kingdom. Accurate predictions of the forces exerted on these devices are needed in order to ascertain the relative efficiencies of the many devices that have been proposed. (See Evans (1981) for a recent review of this subject.)

In many problems involving a large body situated in the ocean, a realistic mathematical model can be achieved by assuming that the fluid is inviscid, incompressible and irrotational, and that surface-tension effects can be ignored. This leads to the classical description of the flow in terms of a velocity potential which satisfies Laplace's equation in the bulk of the fluid. In the context of a linearized theory, the boundary condition at the free surface reduces to a Robin-type condition involving a linear combination of the potential and its normal derivative. Even after making all of these simplifying assumptions we still need to solve a non-trivial boundary-value problem.

For general 3-dimensional bodies a solution can be formulated in terms of integral equations. The usual procedure is to use a 'singular solution', such as a Green function, which satisfies all the conditions of the problem except that on the body itself. When the potential is expressed in terms of a distribution of these 'sources' over the body surface, the problem reduces to one of finding the solution of a 2-dimensional integral equation of the second kind, which gives the requisite source density over the body surface. A detailed investigation of the properties of this integral equation has been given by John (1950), who also established the existence and uniqueness of a solution to the physical problem, subject to the satisfaction by the body of certain geometrical conditions.

Only in a few exceptional cases can this integral equation be solved explicitly. However, with the advent of modern electronic computers with their large storage capacities it is feasible to discretize the body surface into a number of flat 'panels' and so replace the integral equation by a related system of linear algebraic equations whose solution gives approximations to the average source density over each panel. Details of the application of this procedure have been given by Kim (1965); Milgram & Halkyard (1971); Garrison & Chow (1972); Hogben & Standing (1974), and others.

This paper describes an integral equation method which is more appropriate for floating bodies which have a vertical axis of symmetry. For this type of body the problem will be shown to reduce to the solution of 1-dimensional integral equations of the second kind, in which the kernels relate to the potentials of horizontal rings of sources which are distributed over the wetted part of the body surface. It seems reasonable to expect that solutions can be more accurately obtained from these 1-dimensional equations than from the 'full' 2-dimensional equation, since one of the integrations is being performed analytically rather than numerically. Although this is essentially the same approach as that used earlier by Black (1975) and Fenton (1978), the present work differs from that of the previous authors in that more attention is given to finding accurate and efficient means of computing the terms appearing in the kernels of the integral equations.

The ring-source/integral-equation method will only be of practical value if we can find analytic expressions for the ring-source potential which easily lend themselves to numerical computation. Fenton has given an expression for this potential in the form of an infinite series of cylindrical harmonics, see (3.9) below, but this is found to converge only very slowly in the neighbourhood of a vertical cylinder that has the ring on its surface. Thus, for a distribution of rings over the body, one obtains regions of slow convergence throughout the vertical cylinder containing the body – this is often the region of greatest interest in the calculations. Fenton was able to improve the convergence of his infinite series by a process of 'series transformation' and 'removal of singularities', but this considerably increases the algebraic complexity of his final expression for the ring-source potential.

It is possible to derive many alternative expressions for the ring-source potential, but it appears that no single expression is computationally convenient in every situation; the aim of the present paper is to give the most useful of these expressions, so as to provide a comprehensive scheme for the evaluation of the ring-source potential. For example, §3 gives an expression for the potential due to a '$\cos m\theta$' distribution of sources around a horizontal ring as an infinite series of multivalued toroidal harmonics, and this is computationally useful in the neighbourhood of the ring, particularly if the ring lies on, or near, the free surface. This generalizes the result given in a previous paper (Hulme 1981*a*) which considered the case of a uniform distribution of sources, i.e. $m = 0$.

It is well known that the representation of the potential as a surface source distribution will fail at a discrete set of 'irregular' values of the wavenumber K, for which the corresponding interior Dirichlet problem has non-trivial solutions. For an arbitrary 3-dimensional body these values of K are not known in advance, but it is shown in §5 that, for bodies that possess a vertical axis of symmetry, the irregular values of K lie near to the zeros of the Bessel functions J_m, $m = 0, 1, \ldots$.

From a practical viewpoint there are two physically distinct cases of interest: the 'radiation' problem in which waves are produced in the fluid as a result of a forced motion of the body, and the 'diffraction' problem in which waves are incident upon a fixed body and are modified by it. The mathematical formulations of these problems are almost identical and so only that for the radiation problem will be given in detail.

The ring-source/integral-equation method has been used to compute the wave forces exerted on certain specific bodies of revolution, and some of the results are presented in §6.

An alternative method of finding numerical solutions to water wave problems of this type is to use a 'finite-elements' approach, but this will not be discussed here. For a review of this and other methods see Mei (1978).

2. The mathematical formulation of the problem: rings of sources

We will now formulate the radiation problem for a floating body of revolution which has a vertical axis of symmetry. The surrounding fluid is assumed to be inviscid, imcompressible and irrotational, and surface-tension effects will be ignored. These assumptions suggest a description of the flow in terms of a velocity potential $\mathbf{\Phi}(\mathbf{r}, t)$, which must be defined in the region $\mathbb{V}$ exterior to the wetted part of the body surface $\mathbb{S}$. Waves are generated in the fluid due to an oscillatory motion of the body, whose instantaneous velocity will be taken as $\mathbf{V}\cos\omega t$, where $\mathbf{V}$ is some constant vector. The fluid is assumed to have attained a 'steady state' in which its variation with time is also harmonic, and we can write

$$\mathbf{\Phi}(\mathbf{r}, t) = \mathrm{Re}\{\phi(\mathbf{r})\, e^{-i\omega t}\},$$

where ϕ is a complex-valued potential, to be determined. The equation of continuity in the bulk of the fluid is

$$\nabla^2\phi = 0 \quad \text{in } \mathbb{V}, \tag{2.1}$$

where ∇^2 is the 3-dimensional Laplacian operator. We assume that the fluid motion is small enough to allow the use of the linearized free-surface condition

$$K\phi + \frac{\partial\phi}{\partial y} = 0 \quad \text{on} \quad y = 0 \quad \text{outside } \mathbb{S}, \tag{2.2}$$

where $K = \omega^2/g$. The fluid has a uniform depth d and the potential satisfies the fixed, non-porous wall condition

$$\frac{\partial\phi}{\partial y} = 0 \quad \text{on} \quad y = d. \tag{2.3}$$

It seems reasonable to expect that far from the body the potential ϕ resembles that of a radially outgoing wave and so we impose a 'radiation condition' of the form

$$r^{\frac{1}{2}}\left\{\frac{\partial}{\partial r} - ik_0\right\}\phi \to 0 \quad \text{as} \quad r \to \infty, \tag{2.4}$$

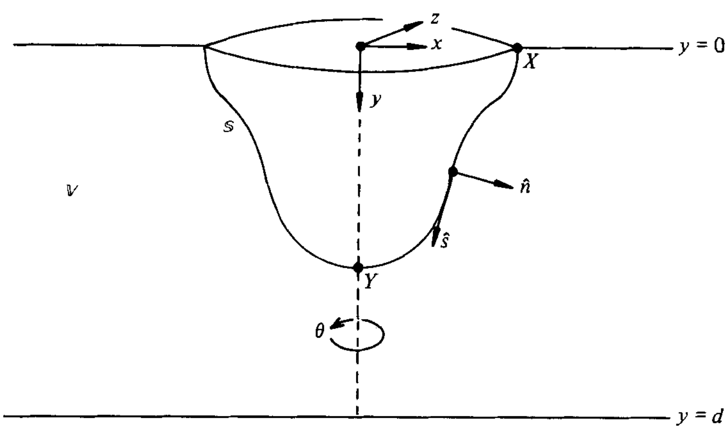

FIGURE 1. Definition sketch.

where $$r^2 = x^2 + z^2$$

and k_0 is the unique, real, positive root of the transcendental equation

$$K = k_0 \tanh k_0 d.$$

The normal fluid velocity on $\mathbb{S}$ is prescribed by the instantaneous velocity $\mathbf{V}\cos\omega t$ of the body. For simplicity we shall choose the directions of the x- and z-axes so that $\mathbf{V}$ can be written as

$$\mathbf{V} = (v_0\mathbf{j} + v_1\mathbf{i}),$$

where v_0, v_1 are constants and $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ is the usual orthonormal set of basis vectors. The boundary condition on the body is thus

$$\frac{\partial\phi}{\partial n} = \mathbf{V}\,.\,\hat{\mathbf{n}} \quad \text{on } \mathbb{S}. \tag{2.5}$$

The equations (2.1)–(2.5) define a boundary-value problem for the unknown potential $\phi(\mathbf{r})$. This can be solved by the introduction of a Green function $G(\mathbf{r};\mathbf{r}')$ which satisfies all the conditions of the problem except that on the body itself, and has the property that

$$\nabla^2 G = -4\pi\delta(\mathbf{r}-\mathbf{r}'),$$

where δ is the Dirac delta-function. Thus $G(\mathbf{r};\mathbf{r}')$ corresponds to the potential at $\mathbf{r}$ due to a R^{-1} type source at $\mathbf{r}'$. John (1950) has given an expression for this Green function in terms of cylindrical polar coordinates (r,θ,y) and this will be used in §3; for the purpose of this section we need only note that $G(\mathbf{r};\mathbf{r}')$ can be expanded in the form

$$G(\mathbf{r};\mathbf{r}') = \sum_{m=0}^{\infty} \epsilon_m g_m(r,y;r',y')\cos m(\theta-\theta'), \tag{2.6}$$

where $$\theta = \tan^{-1}\frac{z}{x}$$

and ϵ_m is Neumann's coefficient ($=1$ when $m=0$, 2 when $m \geqslant 1$).

When solving water-wave problems of this type it is usual to express the potential ϕ as a distribution of sources over the mean wetted surface $\mathbb{S}$, viz

$$\phi(\mathbf{r}) = \iint_{\mathbb{S}} f(\mathbf{r}')\,G(\mathbf{r};\mathbf{r}')\,dS. \tag{2.7}$$

(We adopt the convention that 'primed' variables refer to the surface $\mathbb{S}$.) By construction, this satisfies all the conditions of the problem except that on the body; to satisfy this last condition we must choose the (as yet unknown) source density $f(\mathbf{r}')$ so that

$$-2\pi f(\mathbf{r})+\iint_{\mathbb{S}} f(\mathbf{r}')\frac{\partial}{\partial n}G(\mathbf{r};\mathbf{r}')\,dS = \mathbf{V}.\hat{\mathbf{n}} \quad (\mathbf{r}\in\mathbb{S}) \tag{2.8}$$

and this is valid for *any* surface $\mathbb{S}$.

It is known (see e.g. John 1950) that the integral equation (2.8) always has a unique solution, except at a discrete set of 'irregular' values of K at which the corresponding interior Dirichlet problem has non-trivial solutions. At these irregular values the potential cannot be expressed in terms of a surface distribution of sources alone – dipoles are required as well. Further consideration of these irregular values, with regard to bodies of revolution, is given in §5 of this paper.

We now exploit the fact that the body has a vertial axis of symmetry to rewrite the surface element dS as $dS = r'\,d\theta'\,ds'$, where s' is a measure of arc-length over a cross-section of the surface $\mathbb{S}$. Also, we can express the source density $f(\mathbf{r}')$ as a Fourier cosine series

$$f(\mathbf{r}') = \sum_{m=0}^{\infty} f_m(s')\cos m\theta'.$$

Substituting these forms into (2.7), we deduce that

$$\theta(\mathbf{r}) = \sum_{m=0}^{\infty}\int_0^{s^*} f_m(s')\,\mathscr{R}_m(\mathbf{r};r',y')\,ds',$$

where

$$\mathscr{R}_m(\mathbf{r};r',y') = r'\int_0^{2\pi}\cos m\theta'\,G(\mathbf{r};\mathbf{r}')\,d\theta'$$

and s^* is the arclength XY in figure 1.

We recognize $\mathscr{R}_m(\mathbf{r};r',y')$ to be the potential at $\mathbf{r}$ due to a $\cos m\theta$ distribution of sources around the ring $y=y'$, $r=r'$; henceforth we will refer to $\mathscr{R}_m$ as the mth-order ring-source potential.

Using the expression for G given by (2.6), we deduce that $\mathscr{R}_m$ is of the form

$$\mathscr{R}_m(\mathbf{r};r',y') = R_m(r,y;r',y')\cos m\theta, \tag{2.9}$$

with

$$R_m(r,y;r',y') = 2\pi r' g_m(r,y;r',y'). \tag{2.10}$$

Finally, we deduce that $\phi(\mathbf{r})$ can be expanded as

$$\phi(\mathbf{r}) = \sum_{m=0}^{\infty}\phi_m(r,y)\cos m\theta, \tag{2.11}$$

where

$$\phi_m(r,y) = \int_0^{s^*} f_m(s')\,R_m(r,y;r',y')\,ds', \tag{2.12}$$

and the (as yet unknown) source-density terms $f_m(s')$ satisfy the equations

$$-2\pi f_m(s)+\int_0^{s^*} f_m(s')\frac{\partial}{\partial n}R_m(r,y;r',y')\,ds' = \begin{cases} v_0\cos\xi & (m=0), & (2.13a)\\ v_1\sin\xi & (m=1), & (2.13b)\\ 0 & (m\geqslant 2) & (2.13c)\end{cases}$$

$(\cos\xi = \hat{\mathbf{n}}.\mathbf{j})$.

Thus for a body of revolution the problem can be reduced to the solution of a system of 1-dimensional integral equations of the second kind. In fact, for $m \geqslant 2$ it is clear that we can take

$$f_m(s) \equiv 0,$$

i.e.

$$\phi_m(r, y) \equiv 0 \quad \text{for} \quad m \geqslant 2,$$

and so for the radiation problem we only need solve the integral equations for $m = 0, 1$.

A similar situation arises in the corresponding diffraction problem, which can also be reduced to a system of equations like (2.13), but with different 'right-hand sides'. For the diffraction problem we write the velocity potential $\phi^{\mathrm{d}}(\mathbf{r})$ as the sum of two terms

$$\phi^{\mathrm{d}}(\mathbf{r}) = \phi^{\mathrm{i}}(\mathbf{r}) + \phi^{\mathrm{s}}(\mathbf{r}),$$

where ϕ^{i} is the (known) potential of the incident wave and ϕ^{s} is the potential of the scattered wave, which is to be determined. To uniquely define the solution to this problem we must impose the radiation condition that, far from the body, ϕ^{s} represents a radially outgoing wave. As before, we write ϕ^{i}, ϕ^{s} as Fourier series

$$\phi^{\mathrm{i}}(\mathbf{r}) = \sum_{m=0}^{\infty} \phi^{\mathrm{i}}_m(r, y) \cos m\theta,$$

$$\phi^{\mathrm{s}}(\mathbf{r}) = \sum_{m=0}^{\infty} \phi^{\mathrm{s}}_m(r, y) \cos m\theta,$$

where the ϕ^{s}_m can be expressed in terms of ring-source distributions

$$\phi^{\mathrm{s}}_m(r, y) = \int_0^{s^*} f_m(s')\, R_m(r, y; r', y')\, ds'.$$

The unknown source-density terms $f_m(s')$ are determined by applying the boundary condition on the fixed body, viz

$$\frac{\partial \phi^{\mathrm{d}}}{\partial n} = 0 \quad \text{on } \mathbb{S},$$

i.e.

$$\frac{\partial \phi^{\mathrm{s}}}{\partial n} = -\frac{\partial \phi^{\mathrm{i}}}{\partial n} \quad \text{on } \mathbb{S}.$$

This leads to the integral equations

$$-2\pi f_m(s) + \int_0^{s^*} f_m(s') \frac{\partial}{\partial n} R_m(r, y; r', y')\, ds' = -\frac{\partial \phi^{\mathrm{i}}_m}{\partial n} \quad \text{on } \mathbb{S} \quad (m = 0, 1, \ldots).$$

The solution of the integral equations for $m = 0, 1$ is sufficient for the calculation of all the net forces and moments acting on the body.

We will see presently that the kernels of these 1-dimensional integral equations are rather complicated and we would not expect to find a solution in 'closed form', so it is necessary to adopt some approximate numerical method of solution. A suitable numerical scheme is outlined in §4. Once numerical approximations to f_0 and f_1 have been found it is then a comparatively easy matter to calculate ϕ_0 and ϕ_1, and hence determine the forces and moments exerted on the body.

However, before we can employ such a numerical method of solution we need to have reliable means of computing the values of the kernels of the integral equations (2.13), and this is discussed in §3.

3. The computation of the ring-source potential R_m

Before we discuss the computation of $R_m(r, y; r', y')$ for any values of the parameters r, y, r', y' it is important to remember that the ring-source potential is singular on the ring itself. It can be rigorously established that

$$R_m(r, y; r', y') \sim -\ln[(r-r')^2+(y-y')^2]+O(1) \quad \text{as} \quad (r-r')^2+(y-y')^2 \to 0$$

(see Hulme 1981*b*), although the form of this result is suggested *a priori*, by analogy with the potential due to an infinite line of sources. The presence of this singularity often means that expressions for $R_m(r, y; r', y')$ are only valid in a restricted range ($y > y'$ or $r > r'$, etc.), as we shall see later.

Three different types of expression for R_m will now be presented, and it is expedient to discuss the computational merits of each one separately. It is often useful to have the form of a result in the limiting case of infinite depth (i.e. $d \to \infty$) and so we will use the revised notation

$$R_m(r, y; r', y'|d), \quad R_m(r, y; r', y'|\infty) \quad \text{etc.}$$

wherever this is appropriate.

3.1. *Integral representations*

John (1950) has given an integral representation of the Green function $G(\mathbf{r};\mathbf{r}')$, viz

$$G(\mathbf{r};\mathbf{r}') = \int_0^\infty p(\mu)\, J_0(\mu q)\, d\mu, \tag{3.1}$$

where

$$p(\mu|d) = 2\cosh\mu(d-y_>)\frac{\mu\cosh\mu y_< - K\sinh\mu y_<}{\mu\sinh\mu d - K\cosh\mu d}, \tag{3.2a}$$

$$p(\mu|\infty) = 2e^{-\mu y_>}\frac{\mu\cosh\mu y_< - K\sinh\mu y_<}{\mu - K}, \tag{3.2b}$$

$$q = \{r^2+r'^2-2rr'\cos(\theta-\theta')\}^{\frac{1}{2}},$$

$$y_> = \max\{y, y'\}, \quad y_< = \min\{y, y'\}.$$

The path of integration in (3.1) is indented to run under the simple pole of the integrand at $\mu = k_0$, so that G satisfies the radiation condition at infinity. (For infinite depth, $k_0 = K$.)

Neumann's addition formula for the Bessel function $J_0(\mu q)$ gives

$$J_0(\mu q) = \sum_{m=0}^{\infty} \epsilon_m J_m(\mu r)\, J_m(\mu r')\cos m(\theta-\theta')$$

(see Watson 1944, p. 358), and, substituting this in (3.1) and using (2.6), (2.10), we deduce that

$$R_m(r, y; r', y') = 2\pi r' \int_0^\infty p(\mu)\, J_m(\mu r)\, J_m(\mu r')\, d\mu. \tag{3.3}$$

For large values of μ the above integrand is asymptotic to

$$e^{-\mu|y-y'|} J_m(\mu r)\, J_m(\mu r'), \tag{3.4}$$

and this suggests that we should rewrite (3.3) in the form

$$R_m(r, y; r', y') = 2\pi r' \int_0^\infty e^{-\mu|y-y'|} F(\mu)\, d\mu,$$

where
$$F(\mu) = e^{\mu|y-y'|}p(\mu)\,J_m(\mu r)\,J_m(\mu r') \tag{3.5}$$
$$= O(1) \quad \text{as} \quad \mu \to \infty.$$

The convergence of this integral is clearly governed by the exponential term $e^{-\mu|y-y'|}$ in the integrand, and so we would expect that the representation (3.4) is numerically convenient whenever $|y-y'|$ is not small compared with r and r'.

To approximate the value of R_m successfully we must also take account of the fact that $p(\mu)$ has a simple pole at $\mu = k_0$. The residue at the pole can easily be shown to be
$$\operatorname{res}\{p(\mu), \mu = k_0\} = 2\frac{k_0^2-K^2}{d(k_0^2-K^2)+K}\cosh k_0(d-y)\cosh k_0(d-y') \tag{3.6}$$
$$= \operatorname{res}(k_0), \quad \text{for brevity.}$$

For the purpose of computation, the expression (3.3) should be evaluated as
$$R_m(r,y;r',y') = 2\pi r'\int_0^C \left\{p(\mu)-\frac{\operatorname{res}(k_0)}{\mu-k_0}\right\} J_m(\mu r)\,J_m(\mu r')\,d\mu \tag{3.7a}$$
$$+2\pi r'\operatorname{res}(k_0)\,J_m(k_0 r)\,J_m(k_0 r')\,⨍_0^C \frac{d\mu}{\mu-k_0} \tag{3.7b}$$
$$+2\pi r'\int_C^\infty e^{-\mu|y-y'|}F(\mu)\,d\mu, \tag{3.7c}$$
where $k_0 < C < \infty$. The integrand of (3.7a) is bounded over $[0, C]$ and the integral can be evaluated using any of the standard quadrature rules, care being taken to treat the 'removable' singularity at $\mu = k_0$. The integral in (3.7b) has the explicit value
$$⨍_0^C \frac{d\mu}{\mu-k_0} = \pi i+\ln\left\{\frac{C}{k_0}-1\right\}.$$

The remaining integral (3.7c) is best evaluated by using a Gauss–Laguerre quadrature scheme, which is specially designed for integrals of this form. (Stroud & Secrest (1966) give a comprehensive description of this and all the other standard Gaussian quadrature rules.)

Finally, we remark that this integral representation is not particularly suitable for evaluating $R_m(r,y;r',y')$ in the neighbourhood of the source ring itself, since here $|y-y'|$ is necessarily small and the convergence of the infinite-range integral (3.7c) will be slow.

3.2. *Series of cylindrical harmonics*

The poles of the function $p(\mu)$ occur as the roots of the equation
$$\mu\sinh\mu d = K\cosh\mu d.$$

This has two real roots $\mu = \pm k_0$ and an infinite set of pure-imaginary roots $\mu = \pm ic_n$, $n = 1, 2, 3, \ldots$, where the $\{c_n\}$ satisfy
$$c_n \tan c_n d = -K. \tag{3.8}$$

John (1950) has given an expansion for $p(\mu)$ of the form
$$p(\mu) = A_0\frac{\mu}{\mu^2-k_0^2} + \sum_{n=1}^{\infty} A_n\frac{\mu}{\mu^2+c_n^2},$$

where

$$A_0 = \frac{4(k_0^2 - K^2)}{d(k_0^2 - K^2) + K} \cosh k_0(d-y) \cosh k_0(d-y'),$$

$$A_n = \frac{4(c_n^2 + K^2)}{d(c_n^2 + K^2) - K} \cos c_n(d-y) \cos c_n(d-y').$$

Substituting this expansion into (3.3) and using the known results

$$\int_0^\infty \frac{\mu}{\mu^2 - k_0^2} J_m(\mu r)\, J_m(\mu r')\, d\mu = \tfrac{1}{2}\pi i\, J_m(k_0 r_<)\, H_m^{(1)}(k_0 r_>),$$

$$\int_0^\infty \frac{\mu}{\mu^2 + c_n^2} J_m(\mu r)\, J_m(\mu r')\, d\mu = I_m(c_n r_<)\, K_m(c_n r_>),$$

where $r_> = \max\{r, r'\}$, $r_< = \min\{r, r'\}$ (see Watson 1944, p. 429), we deduce that

$$R_m(r, y; r', y'|d) = A_0 \pi^2 r' i J_m(k_0 r_<)\, H_m^{(1)}(k_0 r_>) + 2\pi r' \sum_{n=1}^{\infty} A_n I_m(c_n r_<)\, K_m(c_n r_>). \tag{3.9}$$

This representation for R_m is the one used by Black (1975), and later by Fenton (1978).

To investigate the convergence of the infinite series (3.9) we first need to know the behaviour of the $\{c_n\}$ for large n, and this follows from (3.8); it is easy to deduce that

$$c_n d \sim n\pi + O\left(\frac{1}{n}\right) \quad \text{as} \quad n \to \infty.$$

Using this and the known asymptotics of the Bessel functions I_m, K_m (see Watson 1944, pp. 202, 203) we see that

$$I_m(c_n r_<)\, K_m(c_n r_>) \sim \frac{d}{2n\pi} (rr')^{-\frac{1}{2}} e^{-n\pi|r-r'|/d} \quad \text{as} \quad n \to \infty,$$

and it follows that the series representation (3.9) will be rapidly convergent whenever $|r - r'|$ is not small compared with the depth d, and so is computationally useful in this range.

The series representation becomes less useful for larger values of d. In fact, in the formal limit $d \to \infty$ the $\{c_n\}$ become infinitely dense on $[0, \infty)$ and the series transforms to an integral representation, viz

$$\begin{aligned} R_m&(r, y; r' y'|\infty) \\ &= 4K\pi^2 r' i\, e^{-K(y+y')} J_m(Kr_<)\, H_m^{(1)}(K_{r\,>}) \\ &\quad + 8r' \int_0^\infty (\nu \cos \nu y - K \sin \nu y)(\nu \cos \nu y' - K \sin \nu y')\, I_m(\nu r_<)\, K_m(\nu r_>) \frac{d\nu}{\nu^2 + K^2}. \end{aligned} \tag{3.10}$$

The infinite-depth result (3.10) can be accurately evaluated, using an adapted Gauss–Laguerre quadrature rule, whenever $|r - r'|$ is not small compared with either y or y'.

Again, we note that neither (3.9) nor (3.10) is useful for evaluating R_m in the neighbourhood of the ring, as $|r - r'|$ will be small in this region.

3.3. *Expressions using toroidal harmonics*

We now consider expressions for R_m which are computationally useful near the ring itself. First, we define 'local' toroidal coordinates (σ, ψ, θ) about the ring $r = r'$,

$y = y'$; suppose $P(r, y, \theta)$ is a point in the fluid and A, B are opposite ends of a diameter of the ring, such that the plane APB contains the y-axis, and B is closest to P. Then

$$\sigma = \ln\left(\frac{AP}{BP}\right) = \tfrac{1}{2}\ln\left(\frac{(r+r')^2+(y-y')^2}{(r-r')^2+(y-y')^2}\right), \tag{3.11a}$$

$$\psi = A\hat{P}B = \cos^{-1}\left(\frac{4r'^2-AP^2-BP^2}{2AP\,.\,BP}\right), \quad \psi \gtrless 0 \quad \text{for} \quad y \gtrless y', \tag{3.11b}$$

$$\theta \quad \text{as for polar coordinates.} \tag{3.11c}$$

Inversely, we have

$$r = \frac{r'\sinh\sigma}{\cosh\sigma-\cos\psi}, \quad y-y' = \frac{r'\sin\psi}{\cosh\sigma-\cos\psi}. \tag{3.12}$$

The surfaces $\sigma = \text{const.}$, $\psi = \text{const.}$, $\theta = \text{const.}$ are mutually orthogonal, and Laplace's equation is known to have quasi-separated solutions in this coordinate system (see Morse & Feshbach 1953).

Now, suppose (σ, ψ, θ) are as defined above and that $(\sigma^*, \psi^*, \theta)$ is another set of toroidal coordinates defined, in a similar way, relative to the 'image ring' at $r = r'$, $y = -y'$. Then it can be shown that (see Hulme 1981 b)

$$R_m(r,y;r',y') = (-1)^m \pi\sqrt{2}\, 4^m \frac{m!}{(2m)!}(\cosh\sigma-\cos\psi)^{\frac{1}{2}} P^m_{-\frac{1}{2}}(\cosh\sigma) \tag{3.13a}$$

$$+(-1)^m \pi\sqrt{2}\, 4^m \frac{m!}{(2m)!}(\cosh\sigma^*-\cos\psi^*)^{\frac{1}{2}} P^m_{-\frac{1}{2}}(\cosh\sigma^*) \tag{3.13b}$$

$$+E_m(r,y;r',y'), \tag{3.13c}$$

where
$$E_m(r,y;r',y'|d) = 2\pi r'\int_0^\infty \frac{\epsilon(\mu)}{\mu\sinh\mu d - K\cosh\mu d} J_m(\mu r)\, J_m(\mu r')\, d\mu, \tag{3.14}$$

$$\epsilon(\mu) = e^{-\mu d}\,[2\mu\cosh\mu y\cosh\mu y' + K\{\cosh\mu(y-y')-\sinh\mu(y+y')\}] + K\, e^{\mu(d-y-y')},$$

and $P^m_{-\frac{1}{2}}$ is an associated Legendre function of the first kind. The terms (3.13 a, b) correspond to the potentials due to rings of sources in an unbounded fluid, at $r = r'$, $y = y'$ (i.e. $\sigma = \infty$) and $r = r'$, $y = -y'$ (i.e. $\sigma^* = \infty$) respectively. These terms involving toroidal harmonics can be successfully evaluated for any values of the parameters $\sigma, \psi, \sigma^*, \psi^*$ by using the known relations between Legendre functions and complete elliptic integrals (see Appendix A for details).

The integral for E_m, given by (3.14), can be evaluated by quadrature methods similar to those outlined in §3.1, and this is practicable in all cases except when $y+y'$ is small compared with r or r'. Thus difficulties arise when evaluating E_m in the neighbourhood of a ring that lies on, or near, the free surface. In this region σ^* is large, and a more useful representation of E_m, for infinite depths, is the following expansion in terms of multivalued toroidal harmonics, as given by Hulme (1981 a):

$$E_m(r,y;r',y'|\infty) = 2\pi(Ka)\sum_{n=1}^{\infty}\{a_n\frac{\partial B}{\partial\nu}(\sigma^*,\psi^*;\nu,m) - b_n\frac{\partial A}{\partial\nu}(\sigma^*,\psi^*;\nu,m)\}$$

$$+\sum_{n=0}^{\infty} u_n\, A(\sigma^*,\psi^*;\nu,m) + \sum_{n=1}^{\infty} v_n\, B(\sigma^*,\psi^*;\nu,m), \tag{3.15}$$

where
$$A(\sigma^*,\psi^*;\nu,m) = (\cosh\sigma^*-\cos\psi^*)^{\frac{1}{2}}\cos\nu\psi^* Q^m_{\nu-\frac{1}{2}}(\cosh\sigma^*),$$

$$B(\sigma^*,\psi^*;\nu,m) = (\cosh\sigma^*-\cosh\psi^*)^{\frac{1}{2}}\sin\nu\psi^* Q^m_{\nu-\frac{1}{2}}(\cosh\sigma^*),$$

and $Q^m_{\nu-\frac{1}{2}}$ denotes the associated Legendre function of the second kind. The coefficients $\{a_n, b_n, u_n, v_n\}$ satisfy simple three-term recurrence relations from which their values may easily be calculated (see appendix B).

The Legendre function $Q^m_{\nu-\frac{1}{2}}(\cosh\sigma^*)$ can be evaluated from the expression

$$\begin{aligned} Q^m_{\nu-\frac{1}{2}}(\cosh\sigma^*) & \\ &= (-1)^m \pi^{\frac{1}{2}} \frac{(\nu+m-\frac{1}{2})!}{\nu!}(1-e^{-2\sigma^*})^m e^{-(\nu+\frac{1}{2})\sigma^*} F(m+\tfrac{1}{2}, \nu+m+\tfrac{1}{2}; \nu-1; e^{-2\sigma^*}) \\ &= O(e^{-(\nu+\frac{1}{2})\sigma^*}) \quad \text{as} \quad \nu\to\infty, \end{aligned} \tag{3.16}$$

where F is the hypergeometric series, and since it can be shown (see Hulme, 1981*b*) that the coefficients in (3.15) only increase at a rate

$$|a_n+ib_n|, |u_n+iv_n| \sim O(\exp[4(Ka)n]^{\frac{1}{2}}), \quad \text{as} \quad n\to\infty \tag{3.17}$$

we deduce that the series in (3.15) converge for $\sigma^* > 0$ and that convergence becomes more rapid as σ^* increases.

Computational experience has shown that the number of terms needed to be taken in (3.15) in order to obtain a desired accuracy depends strongly on σ^* and to a lesser extent on Ka; for fixed Ka the number of terms decreases as σ^* increases, and if σ^* is fixed the number of terms needed increases as Ka increases. This is exactly the behaviour suggested by the asymptotic results (3.16), (3.17). As an indication of the convergence properties of (3.15), experience has shown that for $\sigma^* \gtrsim 1{\cdot}0$ and $0 \leqslant Ka \leqslant 2{\cdot}0$, less than 25 terms in the series are needed to give an absolute error $< 10^{-6}$.

The expression (3.15) is only valid for the infinite-depth case, but the result is also useful when treating finite depths, since we can write

$$E_m(r,y;r',y'|d) = E_m(r,y;r',y'|\infty) + ⨍_0^\infty h(\mu) J_m(\mu r) J_m(\mu r')\, d\mu, \tag{3.18}$$

where
$$h(\mu) = \frac{4\pi r' e^{-\mu d}(\mu\cosh\mu y - K\sinh\mu y)(\mu\cosh\mu y' - K\sinh\mu y')}{(\mu-K)(\mu\sinh\mu d - K\cosh\mu d)}. \tag{3.19}$$

The function $h(\mu)$ decays like

$$h(\mu) \sim O(e^{-\mu(2d-y-y')}),$$

and so if $y+y'$ is small compared with r and r' the integral (3.19) can be evaluated by the quadrature methods previously described.

4. The numerical solution of the integral equations

We now consider a numerical technique for solving the integral equations (2.13). The aim of the method is to 'approximate' the integral equations by related systems of linear algebraic equations, with constant coefficients.

To assist the analysis the integral equations (2.13) will be written in the general form

$$-2\pi f(s) + \int_0^{s^*} f(s') K(s;s')\, ds' = v(s) \quad (0 \leqslant s \leqslant s^*), \tag{4.1}$$

where

$$K(s;s') = \frac{\partial}{\partial n} R(r,y;r',y')$$

and the suffix m has been omitted. (As before s and s' measure arclengths over the body surface.) The kernel $K(s;s')$ of this integral equation is singular at $s = s'$, and it can be shown that, if the wetted surface $\mathbb{S}$, together with its mirror image in $y = 0$, is of the class C'' (i.e. has continuous first and second derivatives), then

$$K(s;s') = O(\ln|s-s'|) \quad \text{as} \quad |s-s'| \to 0. \tag{4.2}$$

We should notice that this geometrical condition implies that the body should meet the free surface at a right-angle – if this is not the case then $K(s;s')$ will have a 'stronger' singularity at this point.

If we assume that the body is of class C'', then (4.2) implies that the integral equation has a weakly singular kernel, and we can follow a method of solution described by Baker (1977). First, we rewrite (4.1) as

$$\{A(s)-2\pi\}f(s)+\int_0^{s^*}\{f(s')-f(s)\}K(s;s')\,ds' = v(s), \tag{4.3}$$

where

$$A(s) = \int_0^{s^*} K(s;s')\,ds'.$$

The integrand of this equation is now finite (and in fact vanishes) at $s = s'$, although it still has an infinite derivative at this point. Let us now approximate the integral by using a quadrature rule:

$$\int_0^{s^*} F(s')\,ds' = \sum_{j=1}^{N} W_j F(s_j)+E(N), \tag{4.4}$$

where

$$0 = s_1 < s_2 < s_3 < \ldots < s_N = s^*$$

is an N-point dissection of $[0, s^*]$, the $\{W_j\}$ are the 'weights' associated with the rule and $E(N)$ denotes the error in its approximation. Using this in (4.3), and putting $s = s_i$, we have that

$$\{A(s_i)-2\pi\}f(s_i)+\sum_{\substack{j=1\\(j\neq i)}}^{N} W_j\{f(s_j)-f(s_i)\}K(s_i;s_j) = v(s_i)-E(N) \quad (i = 1,2,3,\ldots,N).$$

Rearranging the expression gives

$$\Big\{A(s_i)-2\pi-\sum_{\substack{k=1\\(k\neq i)}}^{N} W_k K(s_i;s_k)\Big\}f(s_i)+\sum_{\substack{j=1\\(j\neq i)}}^{N} W_j K(s_i;s_j)f(s_j) = v(s_i)-E(N) \quad (i = 1,2,3,\ldots,N). \tag{4.5}$$

In practice, little would be known about the quantity $E(N)$, but it seems reasonable to assume that, as N increases, $E(N)$ becomes small compared with the other terms in the equation. It is then natural to compare (4.5) with the matrix equation

$$\mathbf{M}\tilde{\mathbf{f}} = \mathbf{v}, \tag{4.6}$$

where the matrix $\mathbf{M} = \{m_{ij}\}$ is given by

$$m_{ii} = -2\pi+A(s_i)-\sum_{\substack{k=1\\(k\neq i)}}^{\infty} W_k K(s_i;s_k),$$

$$m_{ij} = W_j K(s_i;s_j) \quad (i \neq j),$$

$$\mathbf{v}^{\mathrm{T}} = (v(s_1),\ldots,v(s_N)).$$

We would expect that if the solution to (4.6) is

$$\tilde{\mathbf{f}}^{\mathrm{T}} = (\tilde{f}_1, \ldots, \tilde{f}_N)$$

then as $N \to \infty$
$$|\tilde{f}_i - f(s_i)| \to 0 \quad (i = 1, 2, \ldots, N),$$

i.e. the $\tilde{f}_i$ become increasingly better approximations to the values of the $f(s_i)$. Approximations to $f(s)$ at other values of $s\,(\neq s_i)$ can be found by interpolation.

Systems of equations like (4.6) can now be solved using standard computer subroutines. Complications arise when $\mathbf{M}$ is singular (i.e. det $\mathbf{M}$ vanishes) or, more commonly, when $\mathbf{M}$ is 'nearly singular'; this usually corresponds to the situation when we are near an irregular value of the integral equation (see below).

The quadrature rule (4.4) can be adapted to take account of any special geometrical features of the surface $\mathbb{S}$; for example, it is sensible to distribute more of the points $\{s_i\}$ in the regions of greatest curvature. The quadrature scheme used in practice was a composite trapezium rule. In this the points $\{s_i\}$ are suitably distributed over $[0, s^*]$, and the weights $\{W_i\}$ are then simply

$$W_i = \begin{cases} \frac{1}{2}(s_2 - s_1) & (i = 1), \\ \frac{1}{2}(s_{i+1} - s_{i-1}) & (i = 2, 3, \ldots, N-1), \\ \frac{1}{2}(s_N - s_{N-1}) & (i = N). \end{cases}$$

For this rule we would expect that the error term $E(N)$ decays like $1/N^2$, and this gives some measure of the accuracy to which the solution of the integral equation can be found using this method.

Although the numerical method outlined above is crude in terms of its numerical analysis, its very simplicity is an advantage when developing the computer program needed to implement it. Computer experience shows that it can produce good approximations to the solutions of the integral equations, insofar as the accuracies that are obtained are sufficient for most engineering applications. Most of the computational labour goes into evaluating the elements of the matrix $\mathbf{M}$, and so, as a general guide, computer costs increase as N^2. For this reason it is unwise to use unnecessarily large values of N in order to produce unwanted accuracy; in practice values of N between 25 and 40 were usually sufficient to give solutions accurate to three significant figures.

5. The irregular frequencies of floating bodies – a brief discussion

It has been mentioned that the ring-source/integral-equation method fails to produce unique solutions for the source densities f_m whenever the wavenumber K is an eigenvalue of the corresponding interior Dirichlet problem – figure 2, where $\phi = \phi_m(r, y) \cos m\theta$, $m = 0, 1, \ldots$ (see e.g. John 1950). These eigenvalues form the set of 'irregular values' associated with the integral equations (2.13), and we note that they are independent of the fluid depth.

The importance of these irregular values is that we expect the numerical scheme used to solve the integral equations to become 'ill-conditioned' within a narrow range of wavenumbers surrounding each irregular wavenumber. This ill-conditioning was encountered numerically when calculating force coefficients for floating bodies of revolution. The matrix $\mathbf{M}$, used in (4.6), depends only on the number N of surface points used and on Ka, the non-dimensional wavenumber, where a is a typical length dimension of the body. The ill-conditioning arises in the neighbourhood of each

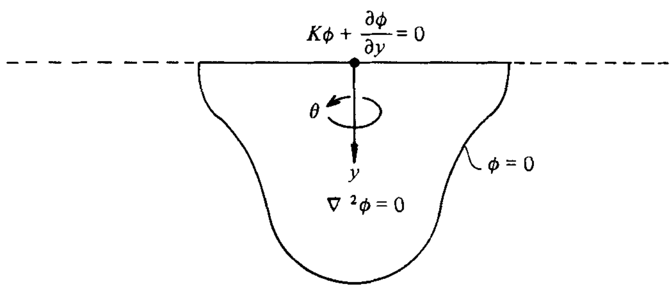

FIGURE 2. The interior Dirichlet problem.

irregular value of Ka, where the matrix $\mathbf{M}$ becomes 'nearly' singular; more precisely, near each irregular value of Ka the quotient

$$Q_m(Ka; N) = \left|\frac{\det\{\mathbf{M}(Ka; N)\}}{\det\{\mathbf{M}(0; N)\}}\right|$$

takes values very close to zero.

Figures 3 and 4 show graphs of Q_m, over $0 \leqslant Ka \leqslant 8{\cdot}0$, for a hemisphere of radius a in heave ($m = 0$) and surge ($m = 1$) respectively. We notice that sharp minima exist over the given range and we expect that each minimum occurs very near to an irregular value of the integral equations; these graphs are quite typical of those obtained for other floating bodies of revolution. Let us denote the nth eigenvalue of the interior Dirichlet problem for a hemisphere (in the modes $m = 0, 1, \ldots$) by α_n^m. The author's calculations suggest that

$$\alpha_1^0 \approx 2{\cdot}56, \quad \alpha_2^0 = 5{\cdot}58,$$

$$\alpha_1^1 \approx 3{\cdot}92, \quad \alpha_2^1 \approx 7{\cdot}06.$$

By a 'minimax' argument of the type used by Garabedian (1964) and more recently by Sayer (1977) we can obtain lower bounds on the α_n^m of the form

$$j_n^m < \alpha_n^m \quad \text{for} \quad m \geqslant 0 \quad \text{and} \quad n \geqslant 1,$$

where the j_n^m are the eigenvalues of the interior Dirichlet problem for a vertical cylinder of radius a. It is a trivial matter to show that the j_n^m are the zeros of Bessel functions, namely

$$J_m(j_n^m) = 0 \quad \text{for} \quad m \geqslant 0 \quad \text{and} \quad n \geqslant 1,$$

and for comparison we note that

$$j_1^0 \approx 2{\cdot}40, \quad j_2^0 \approx 5{\cdot}52,$$

$$j_2^1 \approx 3{\cdot}83, \quad j_2^1 \approx 7{\cdot}02.$$

Moreover, on physical grounds, we would expect that as n increases the j_n^m actually become increasingly good approximations to the eigenvalues α_n^m for any floating body of revolution that meets the free surface normally. (In fact, by closely following a method of Ursell (1974) it can be shown that as $n \to \infty$

$$\alpha_n^m \sim j_n^m + \frac{\lambda^m}{n} + O\left(\frac{1}{n}\right),$$

where the constant λ^m is proportional to the curvature of the surface $\mathscr{S}$ at its points of intersection with the free surface.)

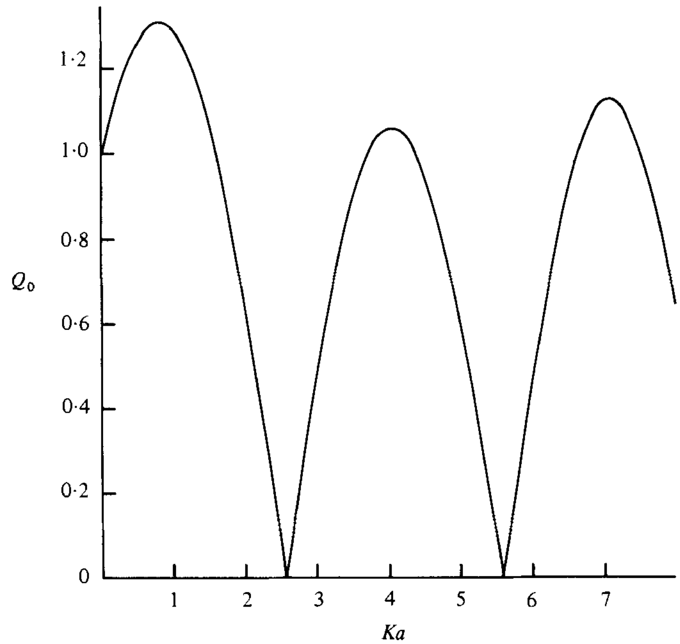

FIGURE 3. The variation of $Q_0(Ka, 26)$ with Ka, for a hemisphere of radius a.

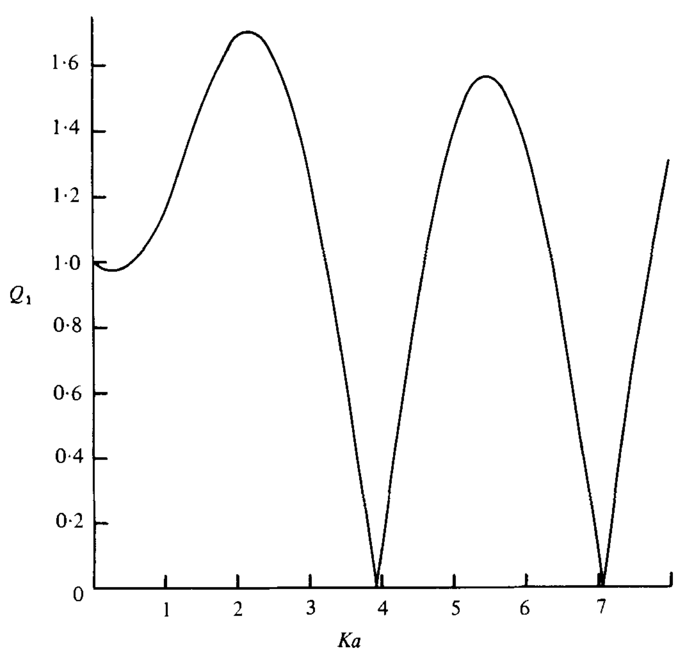

FIGURE 4. The variation of $Q_1(Ka, 26)$ with Ka, for a hemisphere of radius a.

These irregular values have no physical significance and arise only from the representation of the potential as a surface source distribution. If additional singularities are placed inside the body we expect the irregular values to be modified or possibly removed altogether. This is demonstrated in a paper by Ursell (1953) who used this same modification to derive the short-wave asymptotics of a half-immersed circle: it was found that all the higher irregular values were incidentally removed. More recently, Ursell (1981) has shown that similar methods can also be employed to remove the lower irregular values.

In practice, the problem of irregular values is not too difficult to resolve since, for bodies with a vertical axis of symmetry, we have a good idea where the irregular values will occur. (This is certainly not the case for an arbitrary 3-dimensional body, where even the approximate location of the irregular values is not known *a priori*.) If we solve the equation for a value of K that happens to be very close to an irregular value, it becomes obvious that the numerical results are not to be trusted, typically because they do not smoothly join up with the results obtained at values of K slightly away from the irregular value. For such a value of K we can then either use Ursell's method to remove the irregular values or be less demanding and obtain numerical results merely by interpolation, using more reliable results obtained at neighbouring values of K. This last technique is the one adopted by the author in the preparation of his numerical results.

6. Numerical results

Let us consider again the radiation problem as defined in §2, in which the body moves in heave (i.e. performs vertical oscillations), with a velocity $v_0 \cos \omega t\, \mathbf{j}$. The heave force $F_\mathrm{h}\mathbf{j}$ acting on the body is conventionally written as

$$F_\mathrm{h} = v_0\{-A_\mathrm{h}\,\omega \sin \omega t + B_\mathrm{h} \cos \omega t\}. \tag{6.1}$$

The term A_h measures that component of the force that is in phase with the acceleration of the body, and is known as the 'added' (or 'virtual') mass coefficient of the system. The term B_h measures the component of the force in phase with the velocity of the body, and is known as the 'damping' coefficient. Similarly, added-mass and damping coefficients A_s, B_s can be defined for the corresponding surge (i.e. horizontal) motion of the body.

It is usual to present numerical results in terms of dimensionless parameters, and this convention will be followed here: henceforth non-dimensional force coefficients will be denoted by an asterisk. Let V_body be the volume of fluid displaced by the body and ρ be the (uniform) fluid density. Then for the radiation problem, non-dimensional added-mass and damping coefficients A_h^*, B_h^* can be defined by

$$A_\mathrm{h}^* = A_\mathrm{n}/\rho V_\mathrm{body}, \tag{6.2a}$$

$$B_\mathrm{h}^* = B_\mathrm{h}/\omega\rho V_\mathrm{body}, \tag{6.2b}$$

and similarly for A_s^* and B_s^*.

A computer program has been written which numerically implements the ring-source/integral-equation method. In the notation of figure 1, the program needs to be given the shape of the wetted part of the body surface S, in the form

$$r = F(y) \quad (r^2 = x^2 + z^2),$$

and it then proceeds to calculate force coefficients in either the radiation or diffraction problems, for specific choices of $K\,(=\omega^2/g)$. Results for four types of body will be

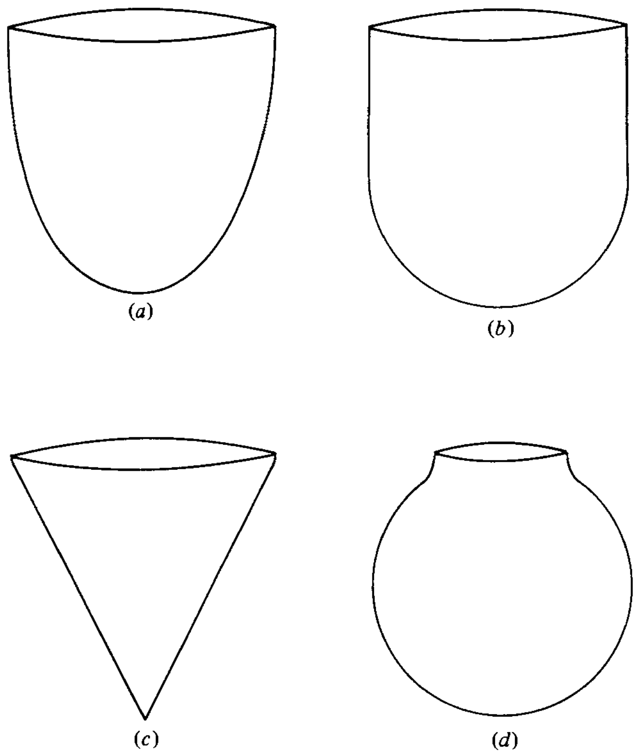

FIGURE 5. (*a*) hemispheroidal body; (*b*) bullet-shaped body; (*c*) cone-shaped body; (*d*) bulb-shaped body.

presented here, and these are depicted in figure 5. The first body (*a*) is hemispheroidal, with its axis of symmetry vertical. The other three bodies (*b*), (*c*) and (*d*) will be referred to (respectively) as the 'bullet', the 'cone' and the 'bulb', for obvious reasons.

The surface of the hemispheroid is given by

$$\frac{r^2}{a^2}+\frac{y^2}{b^2}=1 \quad (0 \leqslant y \leqslant b).$$

Figures 6–9 show the variation of the added-mass and damping coefficients with the non-dimensional wavenumber Ka (for aspect ratios $b/a = 1, \frac{3}{2}, 2$), for fluid of infinite depth. The results seem to be in good agreement with those given previously by Kim (1965) for the aspect ratios $b/a = 1, 2$, care being taken to note that his definitions of A_{h}^*, B_{h}^*, A_{s}^*, B_{s}^* differ by a factor of $\frac{2}{3}(b/a)\pi$ from those used here. The special case of $b/a = 1$ (i.e. a hemisphere) can be formulated 'exactly' in terms of spherical harmonics, following Havelock's classical treatment of the heaving-hemisphere problem (Havelock 1955). This has been done in a recent paper (Hulme 1982), which extends Havelock's method to treat both the heave and surge problems and also tabulates values for the added-mass and damping coefficients for the hemisphere. As a measure of the accuracy obtained here, it was noted that, when a 35×35 matrix was used, the ring-source/integral-equation method produced results for A_{h}^*, B_{h}^*, A_{s}^*, B_{s}^* which agree to 3 or 4 decimal places with the 'exact' values for this problem, over the range $0 \leqslant Ka \leqslant 3$.

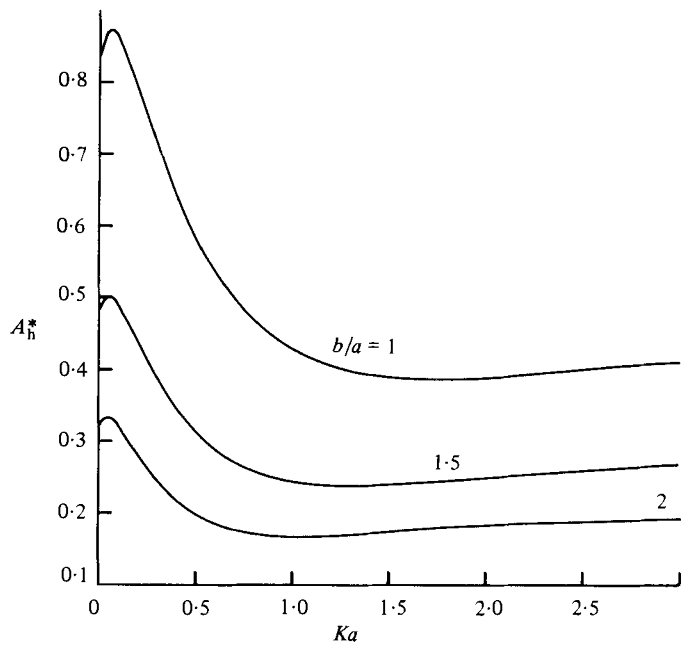

FIGURE 6. The added-mass coefficients of heaving hemispheroids as functions of Ka (infinite depth).

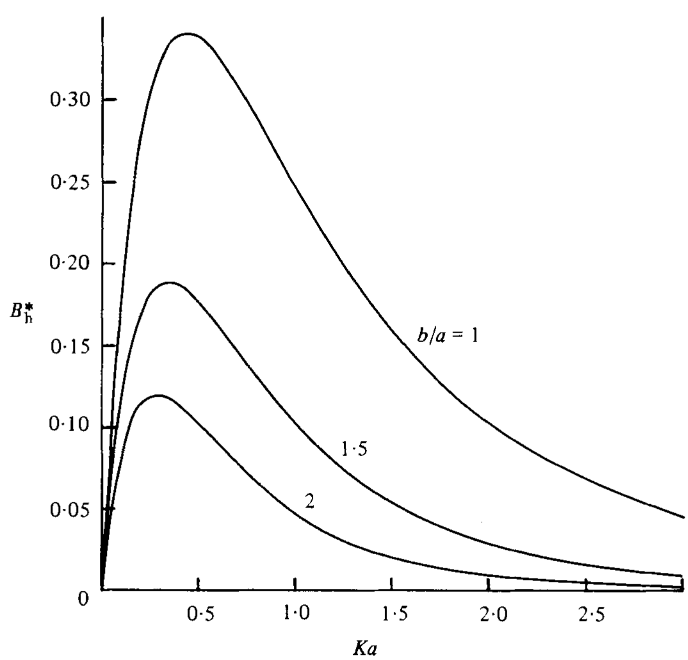

FIGURE 7. The damping coefficients of heaving hemispheroids as functions of Ka (infinite depth).

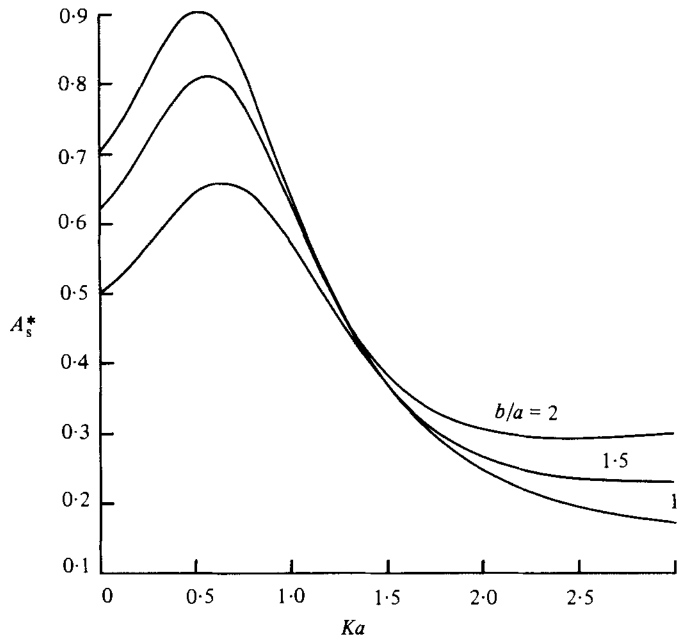

FIGURE 8. The added-mass coefficients of surging hemispheroids as functions of Ka (infinite depth).

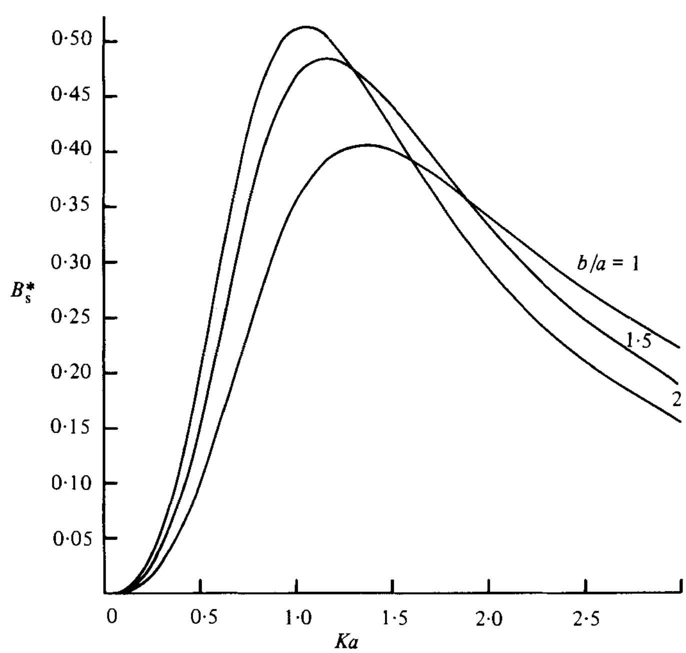

FIGURE 9. The damping coefficients of surging hemispheroids as functions of Ka (infinite depth).

Both the bullet- and the cone-shaped body have recently been discussed in the context of wave-energy extraction from the oceans. Some researchers believe that it may be more practicable to employ arrays of small, isolated point absorbers rather than construct one large, but compact, energy-extraction device. The suitability of using a bullet-shaped body as a point absorber of wave energy has been discussed by Count & Knott (1980), who investigated the heaving motion of such bodies. Figures 10 and 11 show (heave) added-mass and damping coefficients for a 'bullet' whose wetted surface is given by

$$r = \begin{cases} a \quad (0 \leqslant y \leqslant \mu a), \\ \{a^2-(y-\mu a)^2\}^{\frac{1}{2}} \quad (\mu a \leqslant y \leqslant (\mu+1)\,a). \end{cases}$$

The value of μ was chosen to be $\mu = 1{\cdot}055$, so that the relative dimensions of the 'bullet' are the same as those used by Count & Knott. Again, the results shown are for the infinite-depth case.

The efficiency of a cone-shaped body as a point absorber has been investigated by Budal *et al.* (1981). The 'cone' is again assumed to move in heave, and figures 12 and 13 show (heave) added-mass and damping coefficients for a 'cone' whose wetted surface is given by

$$r = \nu a - \tfrac{1}{6}y \quad (0 \leqslant y \leqslant 6\nu a).$$

(Actually, a slight 'rounding' was employed near $y = 0$, to ensure that the body intersected the free surface at a right-angle.) The value of ν was taken as $\nu = 0{\cdot}339$: this is the relevant value for the cone-shaped buoys considered by Budal *et al.* Again, the results shown are for infinite depth.

As yet, all the numerical results that have been presented are for the radiation problem. In contrast, figure 14 shows results for the diffraction problem for a bulbous body. The exact body shape used was

$$r = \begin{cases} \tfrac{1}{2}a + \dfrac{\alpha y^2}{a} - \dfrac{\beta y^3}{a^2} \quad (0 \leqslant y \leqslant a(1-\surd\tfrac{1}{2})), \\ \{y(2a-y)\}^{\frac{1}{2}} \quad (a(1-\surd\tfrac{1}{2}) \leqslant y \leqslant 2a), \end{cases}$$

where

$$\alpha = \frac{3-\surd 2}{\surd 2-1}, \quad \beta = \frac{\surd 8-2}{3-\surd 8}.$$

This surface has continuous first derivatives and meets the free surface at a right-angle. For the diffraction problem we can define a dimensionless heave force coefficient F_{h}^* by

$$F_{\text{h}}^* = (\text{vertical force on body})/\rho g H \Lambda, \tag{6.3}$$

where H is the amplitude of the incident wave, and Λ is the area of the vertical cross-section through the axis of symmetry of the body. Figure 14 shows the variation of F_{h}^* with the dimensionless wavenumber Ka for the given bulb-shaped body in fluid of infinite depth. The behaviour of F_{h}^* in the long-wave region is especially interesting, as the calculations suggest that there is a value of Ka ($\approx 0{\cdot}15$) at which there is no induced vertical force on the body; this phenomenon has been observed experimentally for other bulb-like bodies (see Motora & Kayama 1966).

All the results that have been presented here are for the infinite depth case. The computations were performed on a CDC 7600 computer, and typical 'run times' for one value of Ka, when using a 30×30 discretization matrix, are of the order of 30–35 s. In principle, it is no more difficult to implement the numerical solution of problems involving finite, uniform depths but the computer program would be much more

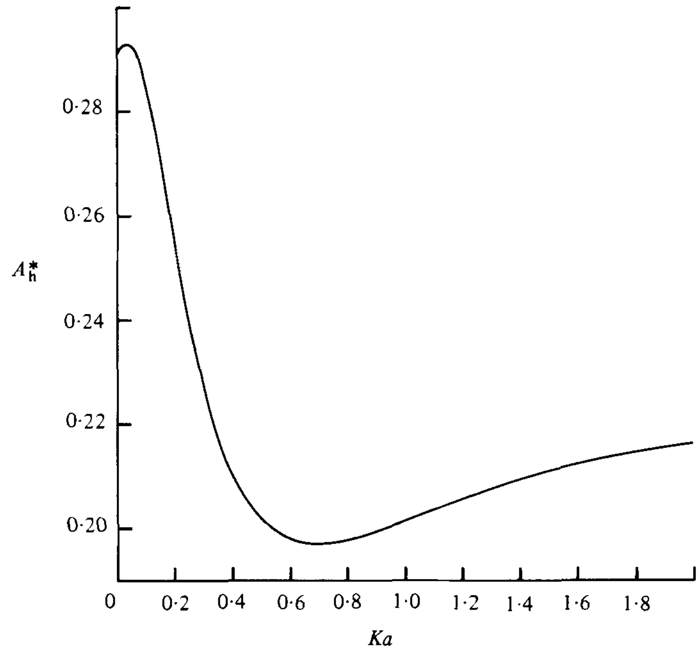

FIGURE 10. The added-mass coefficient of a heaving 'bullet' as a function of *Ka* (infinite depth).

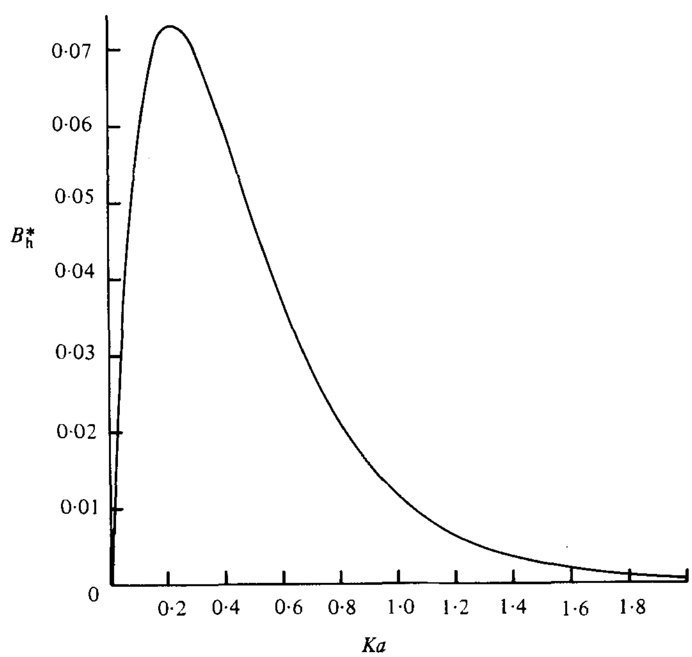

FIGURE 11. The damping coefficient of a heaving 'bullet' as a function of *Ka* (infinite depth).

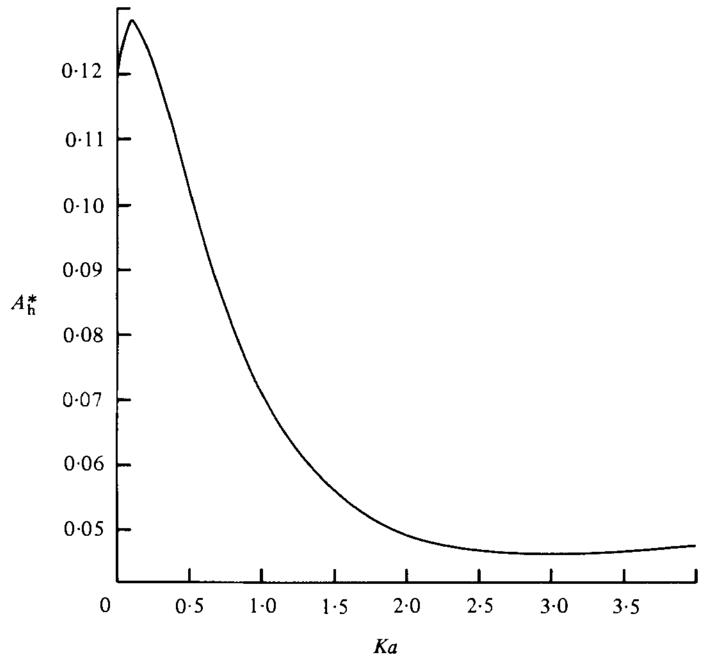

FIGURE 12. The added-mass coefficient of a heaving 'cone' as a function of Ka (infinite depth).

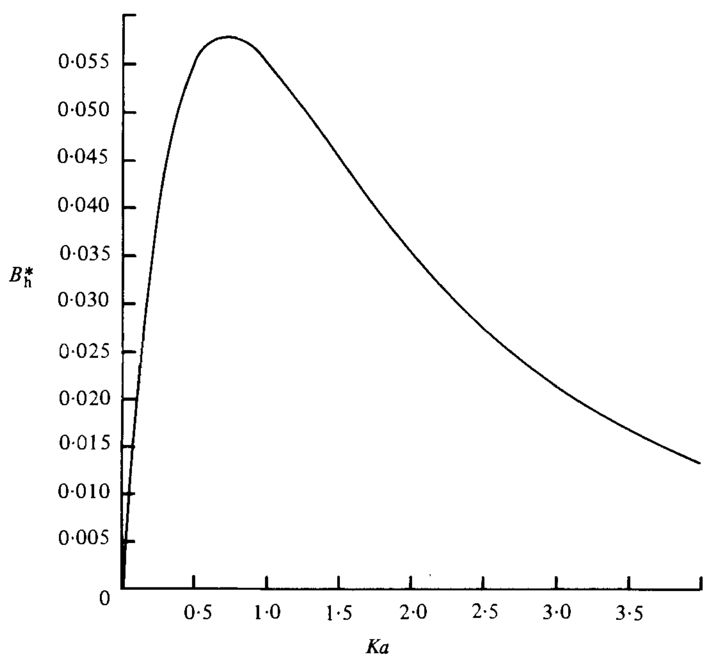

FIGURE 13. The damping coefficient of a heaving 'cone' as a function of Ka (infinite depth).

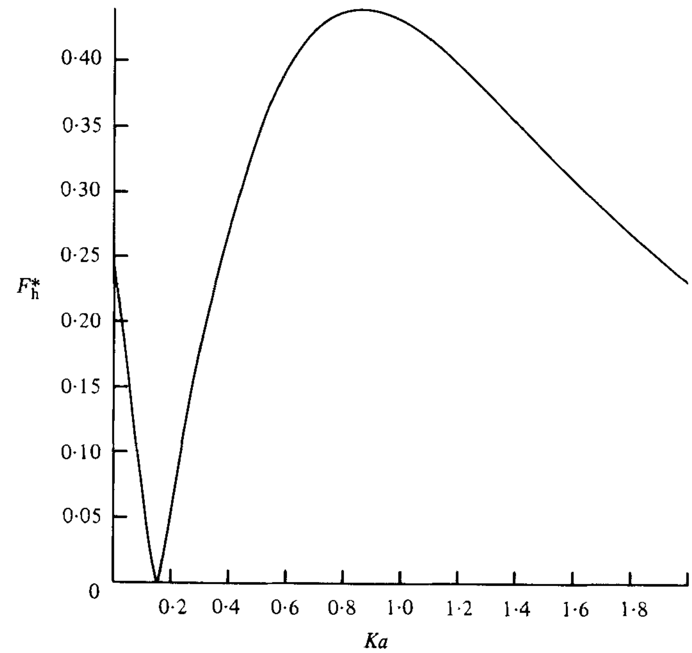

FIGURE 14. The heave force exerted on a bulb-shaped body as a function of *Ka* (infinite depth).

'expensive' to run than its infinite-depth counterpart. This is entirely due to the increased arithmetical complexity of the terms appearing in the kernels of the integral equations.

As a check on the reliability of the numerical results for a certain body geometry it is advisable to compare the results obtained for the force coefficients in the radiation problem with those obtained in the corresponding diffraction problem. If all is well, these should be related by the well-known Haskind relations (see Newman 1962) and the closeness of the agreement will give some measure of the overall accuracy of the calculation.

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Appendix A

The Legendre functions $P^m_{-\frac{1}{2}}(\cosh\sigma)$ can be evaluated for $m = 0, 1$ by using their known relation to the complete elliptic integrals K and E, defined by

$$K(x) = \int_0^{\frac{1}{2}\pi} (1-x^2\sin^2\theta)^{-\frac{1}{2}}d\theta,$$

$$E(x) = \int_0^{\frac{1}{2}\pi} (1-x^2\sin^2\theta)^{+\frac{1}{2}}d\theta.$$

Erdélyi *et al.* (1953) give the results

$$P_{-\frac{1}{2}}(\cosh\sigma) = \frac{2}{\pi}\operatorname{sech}\tfrac{1}{2}\sigma\, K(\tanh\tfrac{1}{2}\sigma), \tag{A 1}$$

$$P^{1}_{-\frac{1}{2}}(\cosh\sigma) = \frac{1}{\pi}e^{\frac{1}{2}\sigma}\operatorname{cosech}\sigma E((1-e^{-2\sigma})^{\frac{1}{2}}) - \frac{1}{\pi}\coth\sigma\operatorname{sech}\tfrac{1}{2}\sigma K(\tanh\tfrac{1}{2}\sigma). \tag{A 2}$$

Abramowitz & Stegun (1970) quote polynomial approximations to $K(x)$ and $E(x)$ which are accurate to 2×10^{-8} over $0 \leqslant x < 1$, care being taken to note the definitions of $K(x)$ and $E(x)$ used above. Thus (A 1) and (A 2) provide an accurate and efficient means of evaluating the Legendre functions $P^{m}_{-\frac{1}{2}}(\cosh\sigma)$, $m = 0, 1$, for all values of σ in the range $0 \leqslant \sigma < \infty$.

Appendix B

The coefficients $\{a_n, b_n, u_n, v_n\}$ appearing in the series expansion (3.15) are most easily evaluated from the three-term recurrence relations which are known to connect them (see Hulme 1981*a*). The 'initial' coefficients a_1, a_2, b_1, b_2 are given explicitly by

$$a_1 = (-1)^m\, 2\sqrt{2}\,\frac{4^{m+1}}{\pi^2}\frac{m!}{(2m+1)!}, \quad a_2 = \frac{4}{2m+3}a_1,$$

$$b_1 = 0, \quad b_2 = -\frac{4(Ka)}{2m+3}a_1,$$

and the 'subsequent' values of $\{a_n, b_n\}_{n\geqslant 3}$ are recursively generated by using the formulae

$$\left.\begin{aligned} a_{n+1} &= \frac{2(Ka)\,b_n + 2na_n - (n-m-\frac{1}{2})\,a_{n-1}}{n+m+\frac{1}{2}}, \\ b_{n+1} &= \frac{-2(Ka)\,a_n + 2nb_n - (n-m-\frac{1}{2})\,b_{n-1}}{n+m+\frac{1}{2}} \end{aligned}\right\} \quad (n \geqslant 2).$$

It is clear that the coefficients a_{N+1}, b_{N+1} depend only on the values of the 'earlier' coefficients $\{a_n, b_n\}_{n\leqslant N}$, and so it is particularly easy to design an algorithm for generating these coefficients on a modern electronic computer.

In a similar way, the 'initial' coefficients u_0, u_1 are specified by

$$u_0 = (-1)^m\sqrt{2}\, 4^{m+1}\frac{m!}{(2m)!}(Ka)⨍_0^\infty J_m^2(\mu)\frac{d\mu}{\mu-Ka}, \tag{B 1}$$

$$u_1 = \frac{2}{2m+1}u_0 + (-1)^m\sqrt{2}\,4^{m+2}\frac{m!}{(2m+1)!}(Ka)\left\{(Ka)⨍_0^\infty J_m(\mu)\dot{J}_m(\mu)\frac{d\mu}{\mu-Ka} - \frac{1}{\epsilon_m}\right\}, \tag{B 2}$$

and by substituting these values into the equations

$$2(Ka)\,u_0 + (m+\tfrac{1}{2})\,v_1 = -2\pi(Ka)\,a_1,$$

$$2(Ka)\,u_1 - 2v_1 + (m+\tfrac{3}{2})\,v_2 = 4\pi(Ka)\frac{2m+1}{2m+3}a_1,$$

$$2(Ka)\,v_1 + (2m-1)\,u_0 + 2u_1 - (m+\tfrac{3}{2})\,u_2 = 8\pi\frac{(Ka)^2}{2m+3}a_1$$

we also obtain u_2, v_1 and v_2. The 'subsequent' coefficients $\{u_n, v_n\}_{n \geqslant 3}$ are then recursively generated by successive substitution in the formulae

$$\left.\begin{aligned} u_{n+1} &= \frac{2(Ka)\,v_n + 2nu_n - (n-m-\frac{1}{2})\,u_{n-1} + \beta_n}{n+m+\frac{1}{2}}, \\ v_{n+1} &= \frac{-2(Ka)\,u_n + 2nv_n - (n-m-\frac{1}{2})\,v_{n-1} - \alpha_n}{n+m+\frac{1}{2}} \end{aligned}\right\} \quad (n \geqslant 2),$$

where

$$\alpha_n = 2\pi(Ka)\{a_{n-1} - 2a_n + a_{n+1}\},$$

$$\beta_n = 2\pi(Ka)\{b_{n-1} - 2b_n + b_{n+1}\}.$$

The integral representations for u_0, u_1 are of little use computationally since the integrands only decay at a rate $O(1/\mu^2)$ as $\mu \rightarrow \infty$. A better method of evaluation comes from considering u_0, u_1 to be functions of Ka and then looking for series expansions of the integrals in powers of Ka. For example, we can rewrite the integral in (B 1) in the form

$$\begin{aligned} \int_0^\infty J_m^2(\mu)\frac{d\mu}{\mu-\xi} &= \fint_0^\infty J_m^2(\mu)\frac{\mu\,d\mu}{\mu^2-\xi^2} + \xi \fint_0^\infty J_m^2(\mu)\frac{d\mu}{\mu^2-\xi^2} + \pi i J_m^2(\xi) \\ &= -\tfrac{1}{2}\pi\{J_m(\xi)\,Y_m(\xi) - 2iJ_m^2(\xi)\} + \xi \fint_0^\infty J_m^2(\mu)\frac{d\mu}{\mu^2-\xi^2} \end{aligned} \qquad \text{(B 3)}$$

(see Watson 1944, p. 429), where

$$\xi = Ka$$

and $\fint$ denotes that the principal value of the integral is to be taken. The integral appearing in (B 3) can be treated by taking its Mellin transform with respect to ξ, re-arranging the order of the integrations, and then using the Mellin inversion theorem to express the original integral as a power series in ξ. The analysis is quite straightforward, but rather laborious, and so we will just state the final result:

$$\begin{aligned} \fint_0^\infty J_m^2(\mu)\frac{d\mu}{\mu-\xi} = -\frac{\pi}{2}\bigg\{ & J_m(\xi)\,Y_m(\xi) - 2iJ_m^2(\xi) + (-1)^m \\ & \times \sum_{n=0}^{\infty}(-1)^n\left(\frac{\xi}{2}\right)^{2n+1}\frac{(2n+1)!}{[(n+\frac{1}{2})!]^2\,(n+m+\frac{1}{2})!\,(n-m+\frac{1}{2})!}\bigg\}. \end{aligned} \qquad \text{(B 4)}$$

The integral in (B 2) can also be expressed in terms of an infinite series by using the relation

$$\fint_0^\infty J_m(\mu)\,\dot{J}_m(\mu)\frac{d\mu}{\mu-\xi} = \frac{1}{2}\frac{d}{d\xi}\fint_0^\infty J_m^2(\mu)\frac{d\mu}{\mu-\xi} + \frac{\delta_{m0}}{2\xi},$$

where

$$\delta_{m0} = \begin{cases} 1 & \text{if} \quad m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

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